# Stat 155 Lecture 6 Notes

Daniel Raban

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## 1 Domination and the Principle of Indifference

### 1.1 Domination by multiple rows or columns

Recall the concept of dominated rows or columns in a payoff matrix from last lecture.

**Definition 1.1.** A pure strategy $e_j$ for player 2 is *dominated* by $e_{i'}$ in the payoff matrix $A$ if for all $i \in \{1, \dots, m\}$, $a_{i,j} \leq a_{i,j'}$.

We can extend this idea to include comparisons with multiple columns.

**Definition 1.2.** A pure strategy $e_j$ for player 2 is *dominated* by columns $e_{j_1}, \dots, e_{j_k}$ in the payoff matrix $A$ if there is a convex combination $y \in \Delta_n$ with $y_j = 0$ and $\{\ell : y_\ell \neq 0\} = \{j_1, \dots, j_k\}$ such that, for all $i \in \{1, \dots, m\}$,

$$a_{i,j} \geq \sum_{\ell=1}^{n} a_{i,\ell} y_\ell.$$

**Theorem 1.1.** *If a pure strategy $e_j$ is dominated by columns $e_{j_1}, \dots, e_{j_k}$, then we can remove column $j$ from the matrix; i.e. there is an optimal strategy for Player 2 that sets $y_j = 0$.*

*Proof.* Let $\tilde{x} \in \Delta_m$ and $\tilde{y} \in \Delta_n$. Then

$$\begin{aligned}
\tilde{x}^\top A \tilde{y} &= \sum_{\ell=1}^{n} \sum_{i=1}^{m} \tilde{x}_i a_{i,\ell} \tilde{y}_\ell \\
&= \sum_{\ell \in \{1,\dots,n\} \setminus \{j\}} \sum_{i=1}^{m} \tilde{x}_i a_{i,j} \tilde{y}_\ell + \sum_{i=1}^{m} \tilde{x}_i a_{i,\ell} \tilde{y}_j \\
&\geq \sum_{\ell \in \{1,\dots,n\} \setminus \{j\}} \sum_{i=1}^{m} \tilde{x}_i a_{i,j} \tilde{y}_\ell + \sum_{i=1}^{m} \tilde{x}_i \left( \sum_{s=1}^{k} a_{i,j_s} y_{j_s} \right) \tilde{y}_j
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell \in \{1,\ldots,n\} \setminus \{j\}} \sum_{i=1}^{m} \tilde{x}_i a_{i,j} \tilde{y}_\ell + \sum_{s=1}^{k} \sum_{i=1}^{m} \tilde{x}_i a_{i,j_s} (y_{j_s} \tilde{y}_j + y_{j_s}) \\
&= \tilde{x}^\top A \tilde{\tilde{y}},
\end{aligned}
$$

where

$$
\tilde{\tilde{y}} = \begin{cases} \tilde{y}_I & \ell \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_k, j\} \\ 0 & \ell = j \\ y_{j_s} \tilde{y}_j + y_{j_s} & \ell = j_s, s \in \{1, \ldots, k\}. \quad \square \end{cases}
$$

The same holds for dominated columns.

## 1.2 The principle of indifference

We've seen a few examples where the optimal mixed strategy for one player leads to a best response from the other that is indifferent between actions. This is a general principle.

**Theorem 1.2.** *Suppose a game with payoff matrix $A \in \mathbb{R}^{m \times n}$ has value $V$. If $x \in \Delta_m$ and $y \in \Delta_n$ are optimal strategies for Players 1 and 2, then*

$$
\sum_{\ell=1}^{m} x_\ell a_{\ell,j} \geq V \quad \forall j, \qquad \sum_{\ell=1}^{n} y_\ell a_{i,\ell} \geq V \quad \forall i,
$$

$$
\sum_{\ell=1}^{m} x_\ell a_{\ell,j} = V \quad \text{if } y_j > 0, \qquad \sum_{\ell=1}^{n} y_\ell a_{i,\ell} = V \quad \text{if } x_i > 0.
$$

This means that if one player is playing optimally, any action that has positive weight in the other player's optimal mixed strategy is a suitable response. It implies that any mixture of these "active actions" is a suitable response.

*Proof.* To prove the two inequalities, note that

$$
V = \min_{y' \in \Delta_n} x^\top A y' \leq x^\top A e_j = \sum_{\ell=1}^{m} x_\ell a_{\ell,j},
$$

$$
V = \max_{x' \in \Delta_m} (x')^\top A y \geq e_i^\top A y = \sum_{\ell=1}^{n} x_\ell a_{i,\ell}.
$$

Recalling that $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j = 1$, the inequalities give us

$$
V = \sum_{j=1}^{n} V y_j \leq \sum_{j=1}^{n} \sum_{i=1}^{m} x_i a_{i,j} y_j \leq \sum_{i=1}^{m} V x_i = V.
$$

If either of the stated equalities did not hold, then we would have strict inequalities here, implying that $V < V$. $\square$

## 1.3 Using the principle of indifference

Suppose we have a payoff matrix $A$, and we suspect that an optimal strategy for Player 1 has certain components positive, say $x_1 > 0, x_3 > 0$. Then we can solve the corresponding "indifference equalities" to find $y$, say:

$$\sum_{\ell=1}^{n} a_{1,\ell} y_\ell = V, \quad \sum_{\ell=1}^{n} a_{3,\ell} y_\ell = V$$

**Example 1.1.** Recall the game Plus One with payoff matrix

$$\begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n-1 & n \\
\hline
1 & 0 & -1 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 \\
2 & 1 & 0 & -1 & 2 & 2 & 2 & \cdots & 2 & 2 \\
3 & -2 & 1 & 0 & -1 & 2 & 2 & \cdots & 2 & 2 \\
4 & -2 & -2 & 1 & 0 & -1 & 2 & \cdots & 2 & 2 \\
5 & -2 & -2 & -2 & 1 & 0 & -1 & \cdots & 2 & 2 \\
6 & -2 & -2 & -2 & -2 & 1 & 0 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
n-1 & -2 & -2 & -2 & -2 & -2 & -2 & \ddots & 0 & -1 \\
n & -2 & -2 & -2 & -2 & -2 & -2 & \cdots & 1 & 0
\end{array}$$

and reduced (after removing dominated rows and columns) payoff matrix

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

We suspect that $x_1, x_2, x_3 > 0$, so we solve

$$Ay = \begin{pmatrix} V \\ V \\ V \end{pmatrix}$$

to get that

$$y = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}, \quad V = 0.$$